Energy transport and chaos in a one-dimensional disordered nonlinear stub lattice model

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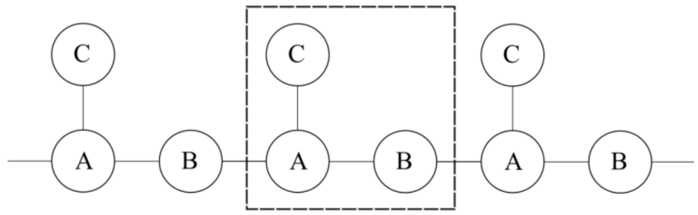
Outline

- **The one-dimensional disordered nonlinear stub lattice model**
  - ✓ **Linear system (flat frequency band, band gaps)**
  - ✓ **Different dynamical behaviors**

- **Numerical results**
  - ✓ **Representative cases of the various dynamical regimes**
  - ✓ **Global investigation of the parameter space**
  - ✓ **The effect of frequency gaps**

- **Summary**

The stub lattice model



Each one of the N unit cells contains three sites labeled A, B and C [Flach et al., EPL (2014); Luck, J. Phys. A (2019)].

$$H = \sum_{n=1}^{N} \left[\varepsilon_n^{(A)} \left|\psi_n^{(A)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(A)}\right|^4 + \varepsilon_n^{(B)} \left|\psi_n^{(B)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(B)}\right|^4 + \varepsilon_n^{(C)} \left|\psi_n^{(C)}\right|^2 + \frac{\beta}{2}\left|\psi_n^{(C)}\right|^4 \right.$$
$$\left. - \left(\psi_n^{(C)*}\psi_n^{(A)} + \psi_n^{(C)}\psi_n^{(A)*}\right) - \left(\psi_n^{(A)*}\psi_n^{(B)} + \psi_n^{(A)}\psi_n^{(B)*}\right) - \left(\psi_{n+1}^{(A)*}\psi_n^{(B)} + \psi_{n+1}^{(A)}\psi_n^{(B)*}\right) \right]$$

with fixed boundary conditions $\psi_{N+1}^{(A)} = 0$.

β: nonlinearity parameter, W: disorder strength. H: total energy.

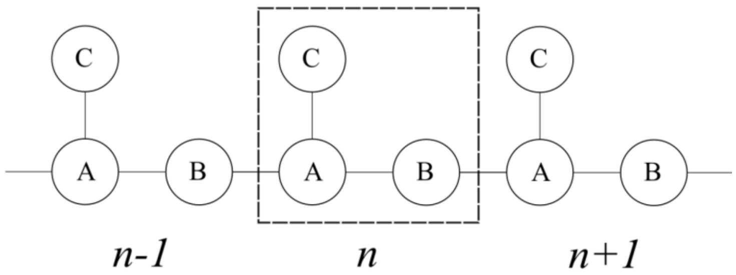
$\varepsilon_n^{(K)}$ (K stands for A, B or C) chosen uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$.

The stub lattice Hamiltonian has formal similarities to the Hamiltonian of the discrete nonlinear Schrödinger equation in one spatial dimension (1D DDNLS):

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - (\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l)$$

The stub lattice model

The canonical transformation $\psi_n^{(K)} = \frac{q_n^{(K)} + i p_n^{(K)}}{\sqrt{2}}$ gives the Hamiltonian the form



$$H = \sum_{n=1}^{N} \Bigg\{ \frac{\varepsilon_n^{(A)}}{2} \left[\left(q_n^{(A)}\right)^2 + \left(p_n^{(A)}\right)^2 \right] + \frac{\beta}{8} \left[\left(q_n^{(A)}\right)^2 + \left(p_n^{(A)}\right)^2 \right]^2 + \frac{\varepsilon_n^{(B)}}{2} \left[\left(q_n^{(B)}\right)^2 + \left(p_n^{(B)}\right)^2 \right]$$

$$+ \frac{\beta}{8} \left[\left(q_n^{(B)}\right)^2 + \left(p_n^{(B)}\right)^2 \right]^2 + \frac{\varepsilon_n^{(C)}}{2} \left[\left(q_n^{(C)}\right)^2 + \left(p_n^{(C)}\right)^2 \right] + \frac{\beta}{8} \left[\left(q_n^{(C)}\right)^2 + \left(p_n^{(C)}\right)^2 \right]^2$$

$$- \left(p_n^{(C)} p_n^{(A)} + q_n^{(C)} q_n^{(A)} \right) - \left(p_n^{(A)} p_n^{(B)} + q_n^{(A)} q_n^{(B)} \right) - \left(p_{n+1}^{(A)} p_n^{(B)} + q_{n+1}^{(A)} q_n^{(B)} \right) \Bigg\}$$

Conserved quantities:

The total energy *H*, and the total norm *S* of the wave packet:

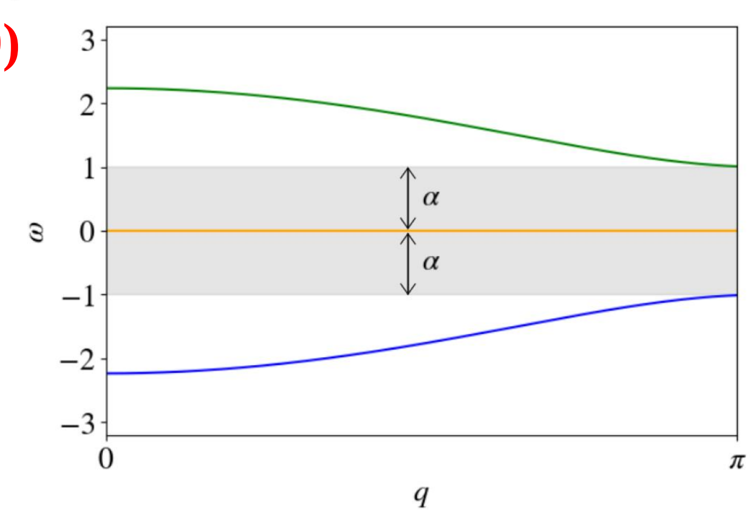
$$S = \sum_{n=1}^{N} \sum_{K} \frac{1}{2} \left[\left(q_n^{(K)}\right)^2 + \left(p_n^{(K)}\right)^2 \right]$$

The linear system

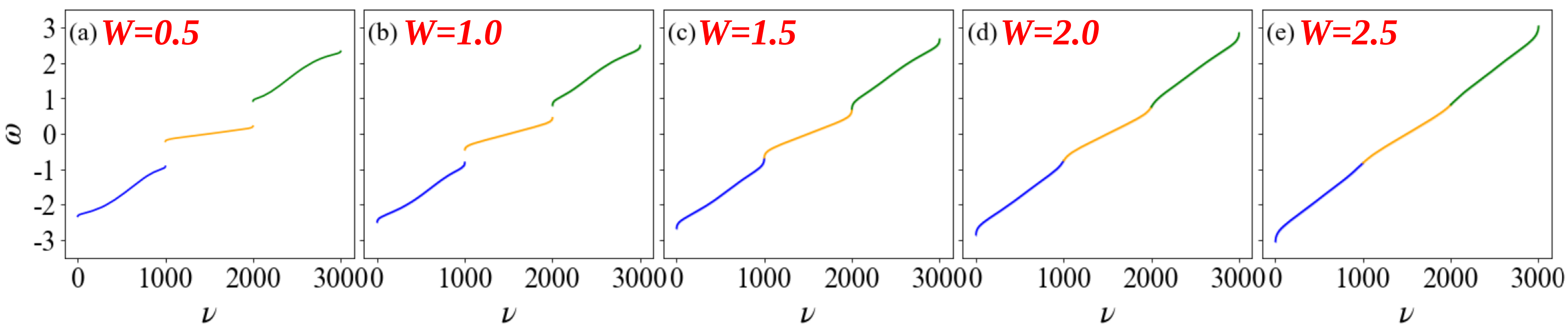
Considering the **linear ($\beta = 0$), ordered ($\varepsilon_n^{(K)}$=0)** stub lattice model and the wave function $\psi_N^{(K)} = U^{(K)} e^{-iqn}\, e^{-i\omega t}$ we obtain the system's dispersion relation which consists of 3 distinct bands (gap size $\alpha = 1$):

$$\omega = \sqrt{3 + 2\ \cos q},\ \omega = 0 \text{ (flat band)},$$

$$\omega = -\sqrt{3 + 2\ \cos q}.$$



Introducing disorder, **the gap disappears for $W \approx 1.58$.** Frequencies (ω) vs. mode number (ν) for $N = 1000$ (averaged over 50 disorder realizations):



Width of frequency spectrum:

$$\Delta = \omega_{max} - \omega_{min} = \left(\frac{W}{2} + \sqrt{5}\right) - \left(-\frac{W}{2} - \sqrt{5}\right) = W + 2\sqrt{5}$$

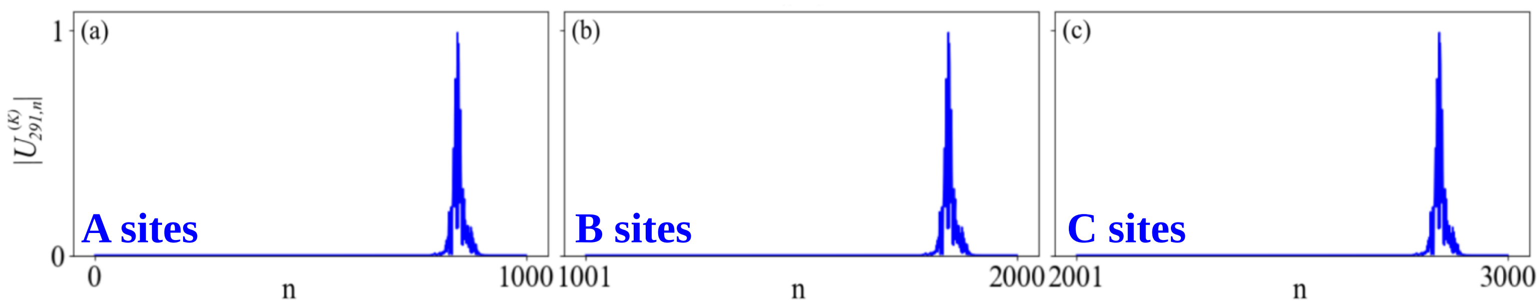
The linear system

Localization volume $V_\nu^{(i)}$ of normal modes (NMs) in ranges $i = 1, 2, 3$ (A, B, C sites respectively):

$$V_\nu^{(i)} = \sqrt{12 m_2^{(\nu,i)}} + 1$$

where $m_2^{(\nu,i)} = \sum_{n=(i-1)N+1}^{iN} \left(n - \bar{n}_\nu^{(i)}\right)^2 \left|U_{\nu,n}^{(i)}\right|^2$ **and** $\bar{n}_\nu^{(i)} = \sum_{n=(i-1)N+1}^{iN} n \left|U_{\nu,n}^{(i)}\right|^2$

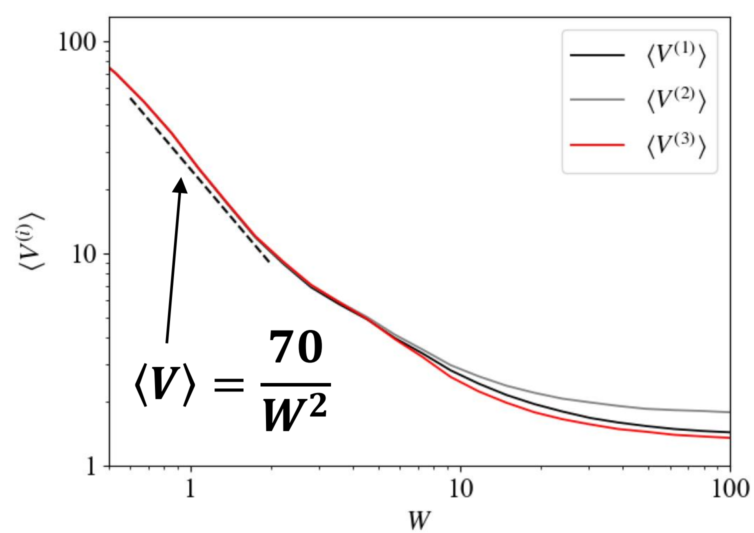
Mode $\nu = 291$ for $W = 1$ ($\beta = 0$).



Considering modes whose mean position is at the central one-third of the 3 ranges, we compute the average (over disorder realizations) $\langle V^{(i)} \rangle$ and then the

average localization volume of NMs

$$\langle V \rangle = \max_i \langle V^{(i)} \rangle, \quad i = 1, 2, 3.$$



Computational techniques

Norm distribution: $\xi_n = \sum_K \frac{\left(q_n^{(K)}\right)^2 + \left(p_n^{(K)}\right)^2}{2S}$

Second moment: $m_2 = \sum_{n=1}^{N} (n - \bar{n}) \xi_n$ **with** $\bar{n} = \sum_{n=1}^{N} n \xi_n$

Participation number: $P = \frac{1}{\sum_{n=1}^{N} \xi_n^2}$

Finite-time maximum Lyapunov exponent (ftMLE): $\Lambda(t) = \frac{1}{t} \ln \left(\frac{||\mathbf{w}(t)||}{||\mathbf{w}(0)||} \right)$

Deviation vector distribution (DVD):

$$\xi_n^D(t) = \frac{\sum_K \left[\left(\delta q_n^{(K)}(t)\right)^2 + \left(\delta p_n^{(K)}(t)\right)^2 \right]}{\sum_{n=1}^{N} \sum_K \left[\left(\delta q_n^{(K)}(t)\right)^2 + \left(\delta p_n^{(K)}(t)\right)^2 \right]}$$

Different Dynamical Regimes

1D DDNLS: Three dynamical regimes [Flach et al., PRL (2009); S. et al., PRE (2009); Flach, Chem. Phys (2010); Laptyeva et al., EPL (2010); Bodyfelt et al., PRE (2011); S. et al., PRL (2013); Senyange et al., PRE (2018)]

Δ: width of the frequency spectrum, $d = \frac{\Delta}{\langle V \rangle}$: average spacing of interacting modes,

$\delta = \beta s$: nonlinear frequency shift.

Weak chaos regime: $\delta < d, m_2 \sim t^{1/3}, P \sim t^{1/6}, \Lambda \sim t^{-0.25}$. NMs are weakly interacting.

Strong chaos regime: $d < \delta < 2, m_2 \sim t^{1/2}, P \sim t^{1/4}, \Lambda \sim t^{-0.3}$. Almost all NMs in the packet are resonantly interacting.

Selftrapping Regime: $\delta > 2$. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, while a small part of the wave packet subdiffuses.

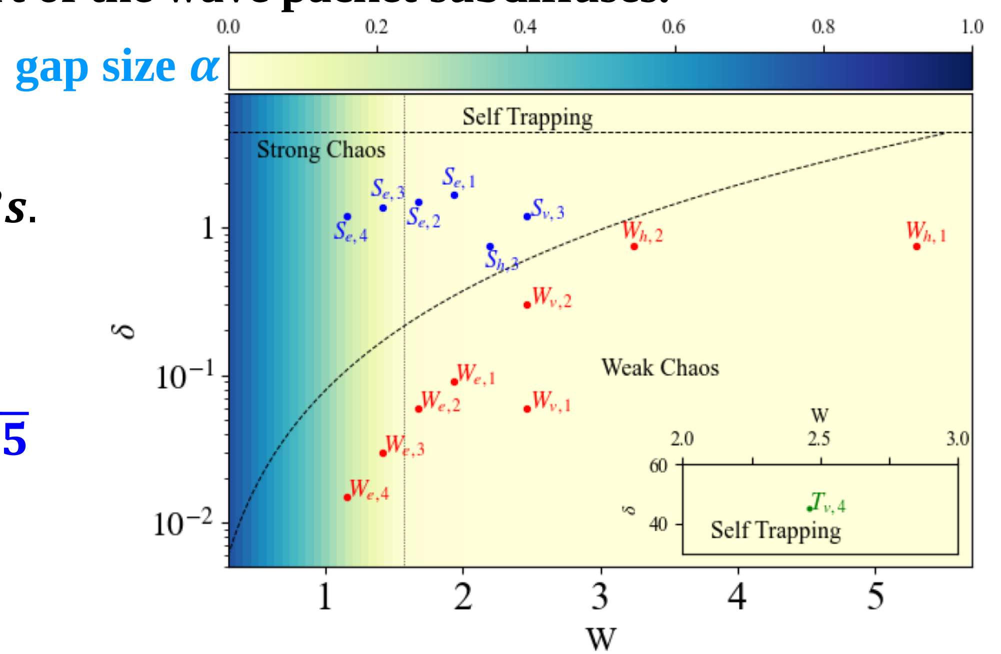
Stub lattice:

$$\Delta = W + 2\sqrt{5},\ d = \frac{\Delta}{\langle V \rangle} = \frac{W^2(W+2\sqrt{5})}{70},\ \delta = \beta s.$$

Weak chaos regime: $\delta < \frac{W^2(W+2\sqrt{5})}{70}$

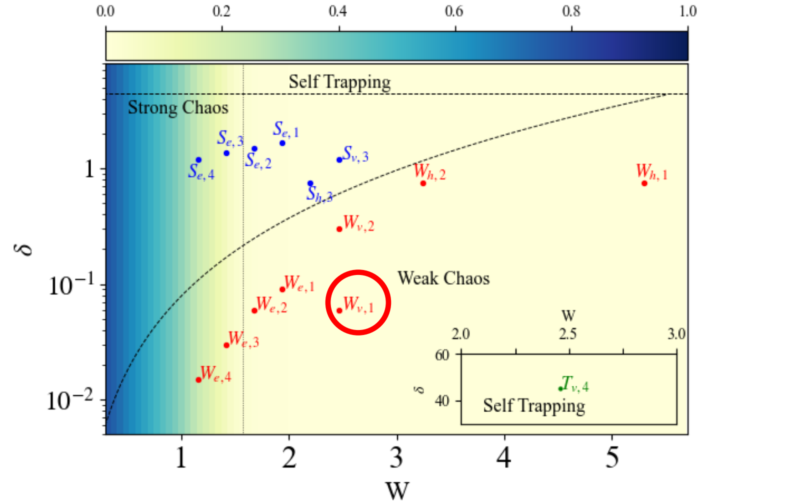
Strong chaos regime: $\frac{W^2(W+2\sqrt{5})}{70} < \delta < 2\sqrt{5}$

Selftrapping Regime: $\delta > 2\sqrt{5}$



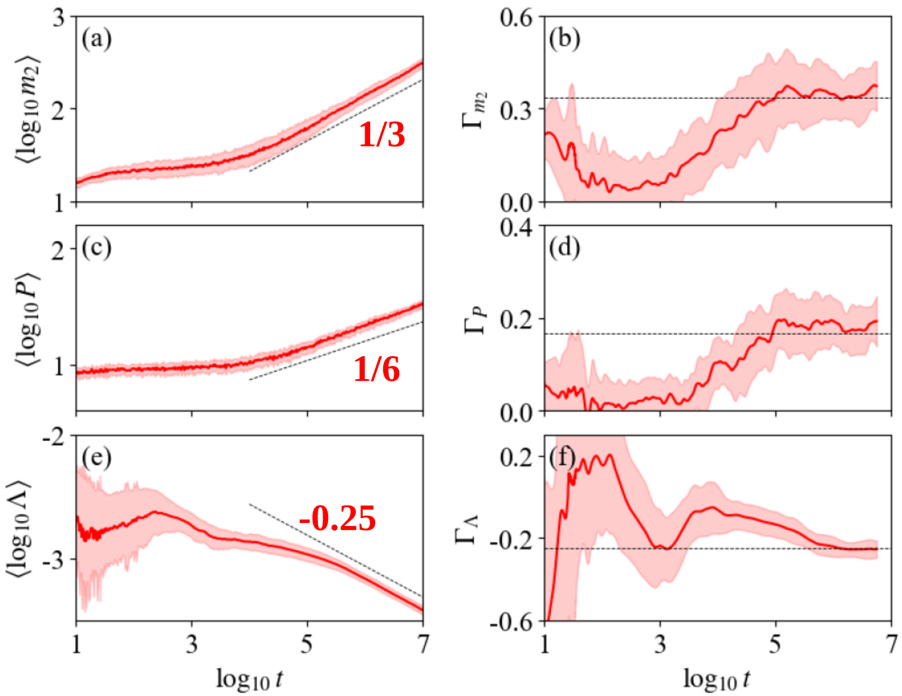
Weak chaos (representative case)

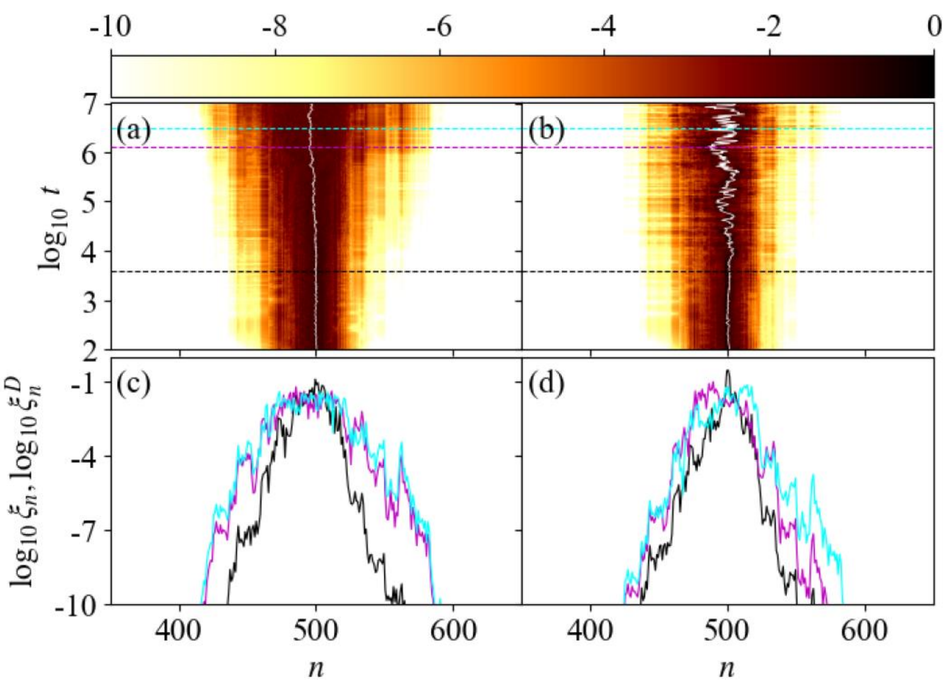
$\boldsymbol{\beta = 0.02, W = 2.46, L = 12, N = 1001}$



Typically, we present the time evolution of a quantity $\langle R \rangle$ in log-log scale and numerically estimate the related rate of change as:

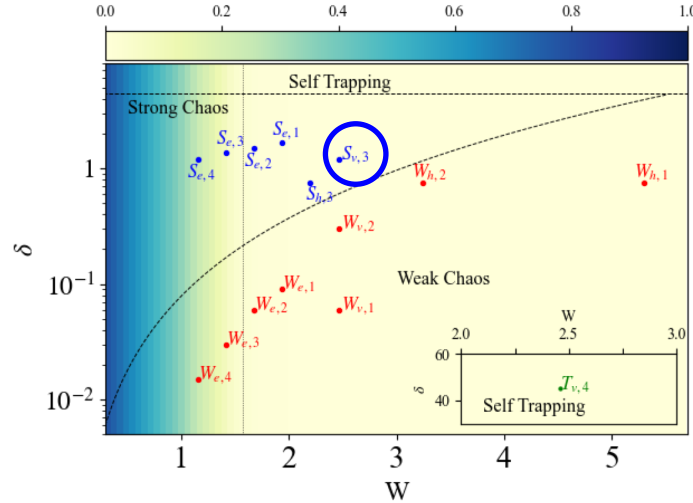
$$\Gamma_R(\log_{10} t) = \frac{d\langle \log_{10} R \rangle}{d \log_{10} t}$$

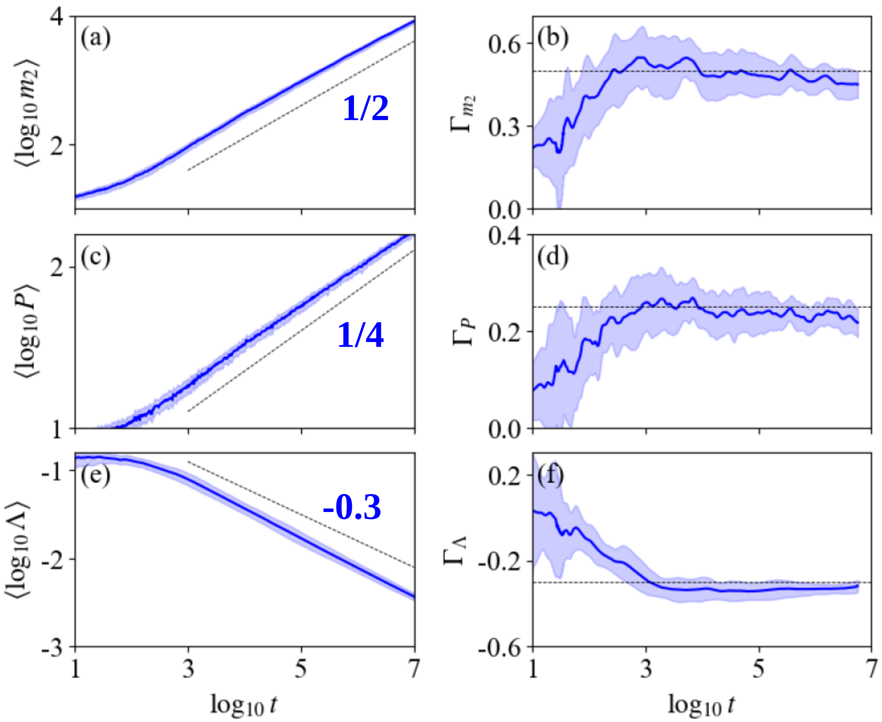


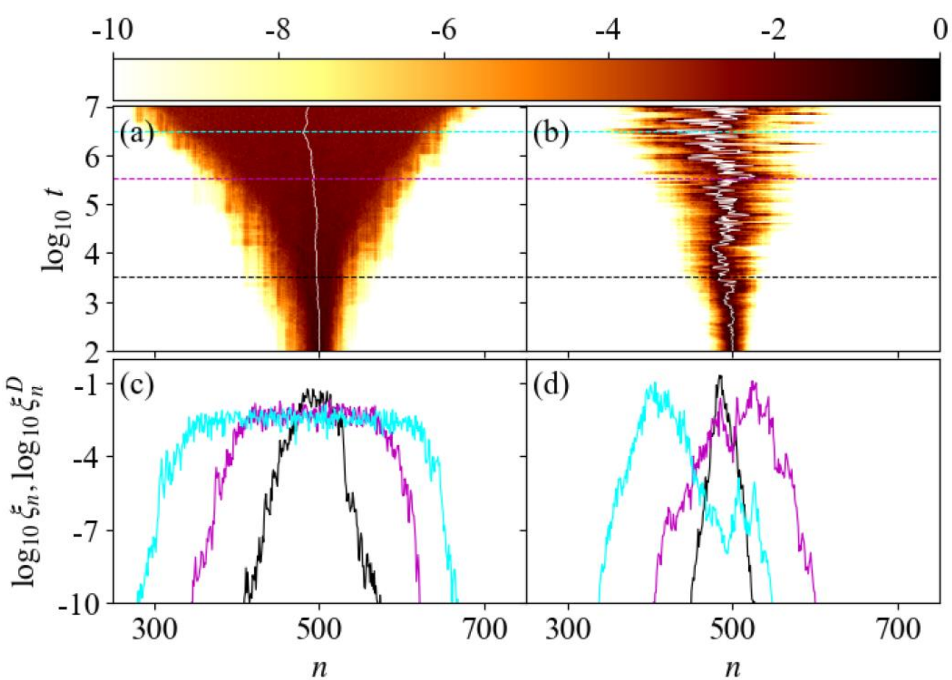


Strong chaos (representative case)

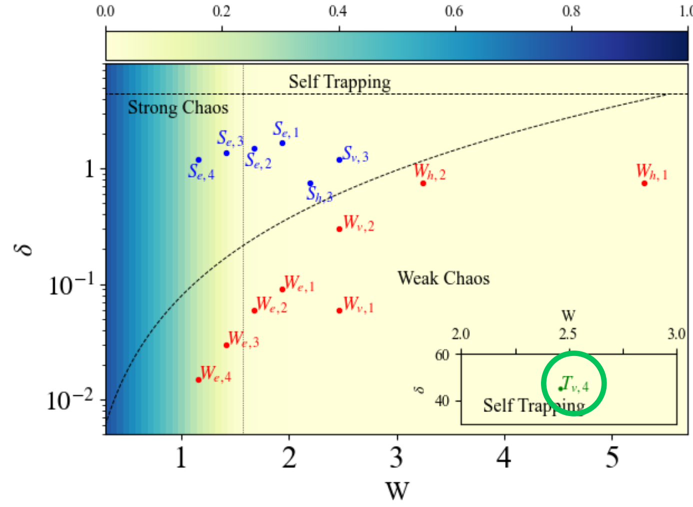
$\beta = 0.4, W = 2.46, L = 12, N = 1001$



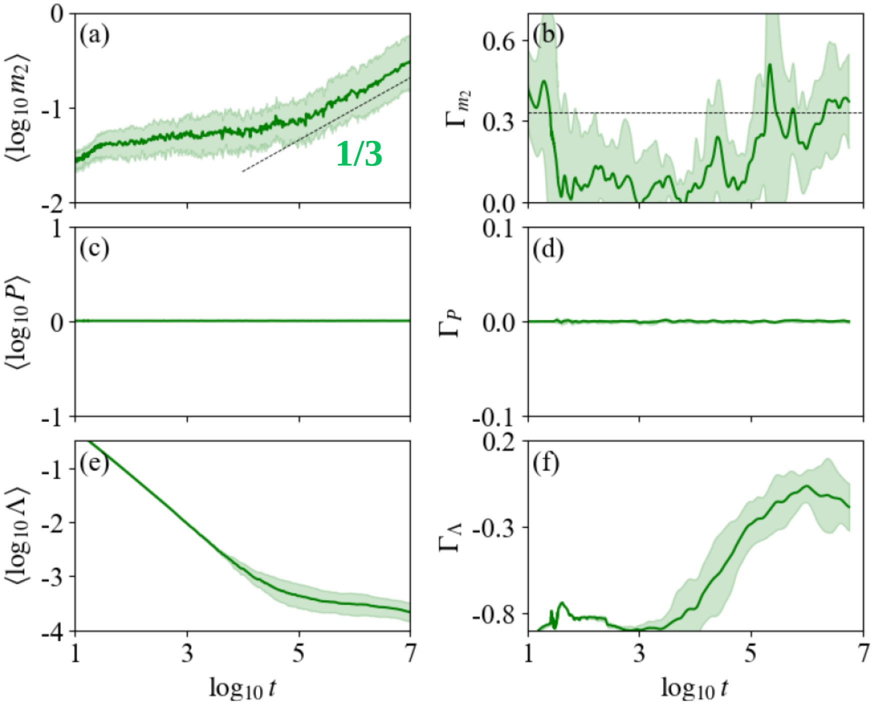


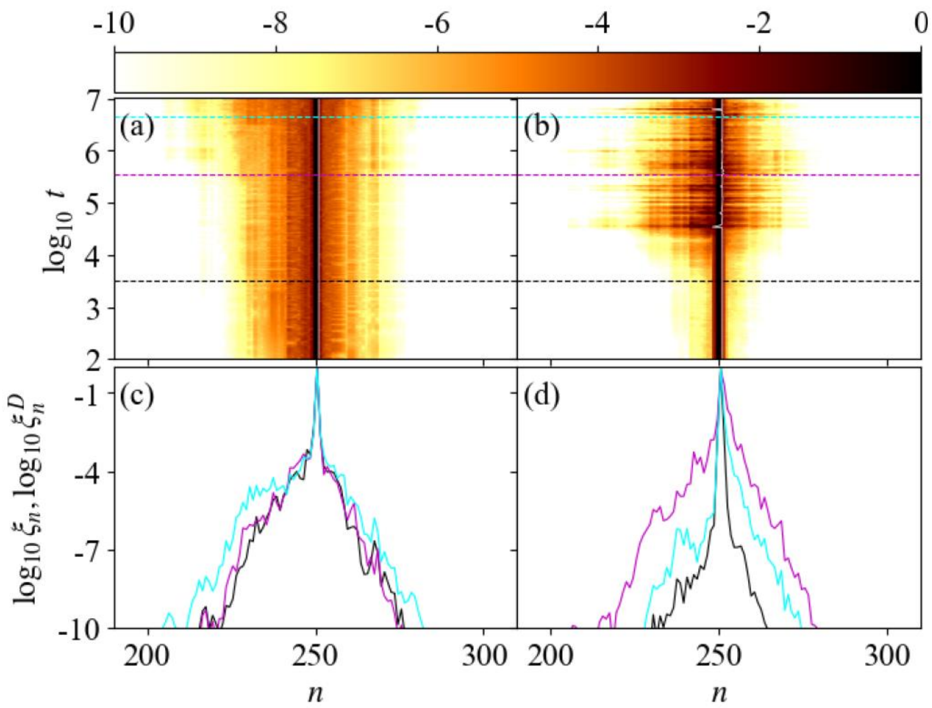


Selftrapping (representative case)



$\beta = 15, W = 2.46, L = 1, N = 501$



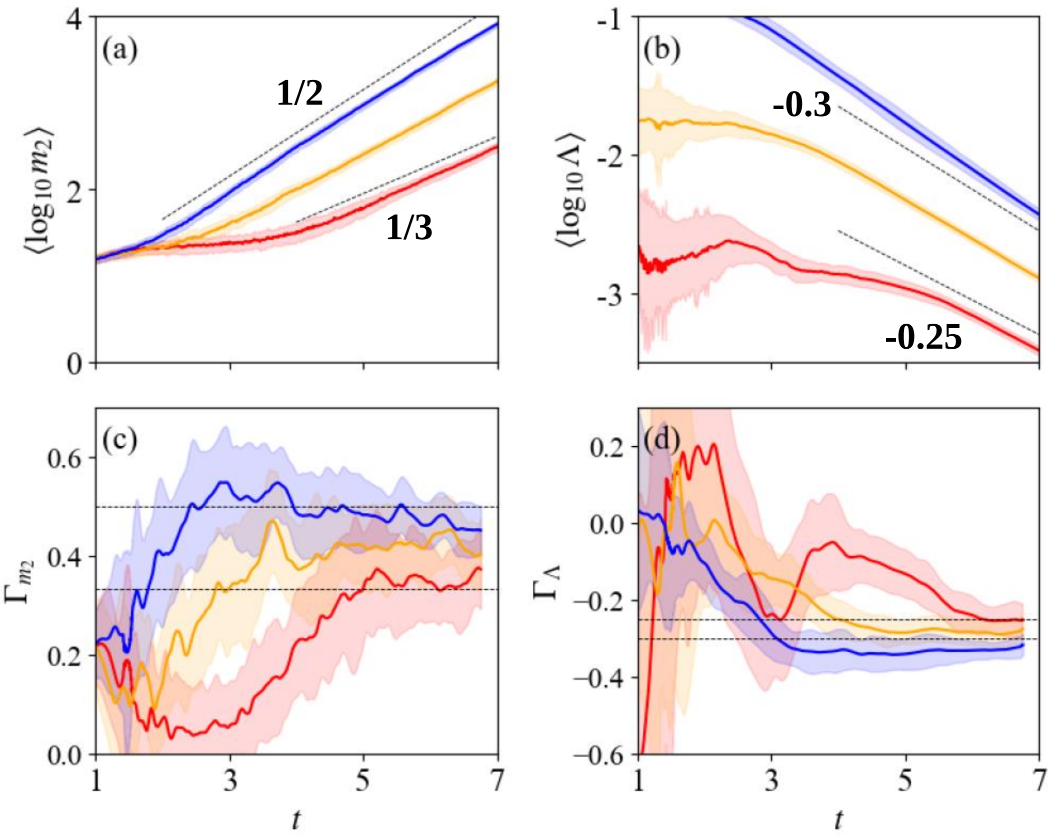


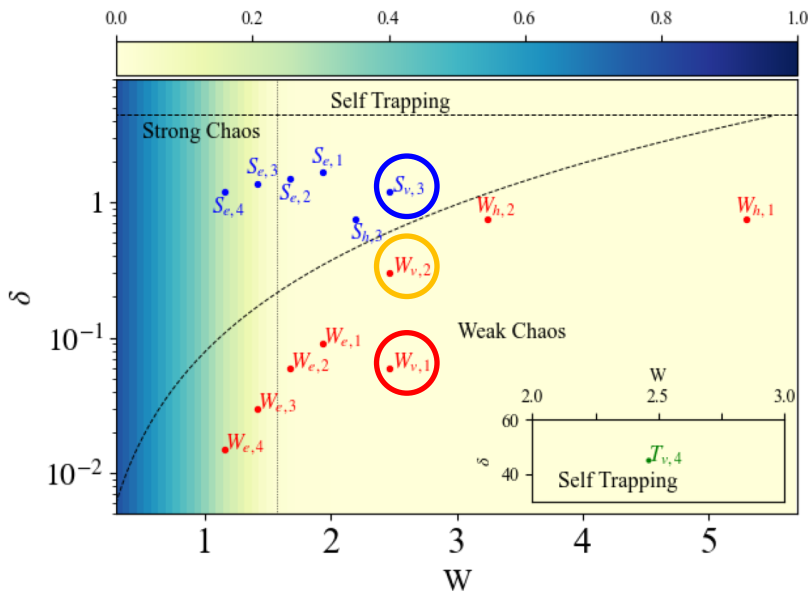
Global investigation I

$\boldsymbol{\beta = 0.02, W = 2.46, L = 12, N = 1001}$

$\boldsymbol{\beta = 0.045, W = 2.46, L = 12, N = 1001}$

$\boldsymbol{\beta = 0.4, W = 2.46, L = 12, N = 1001}$



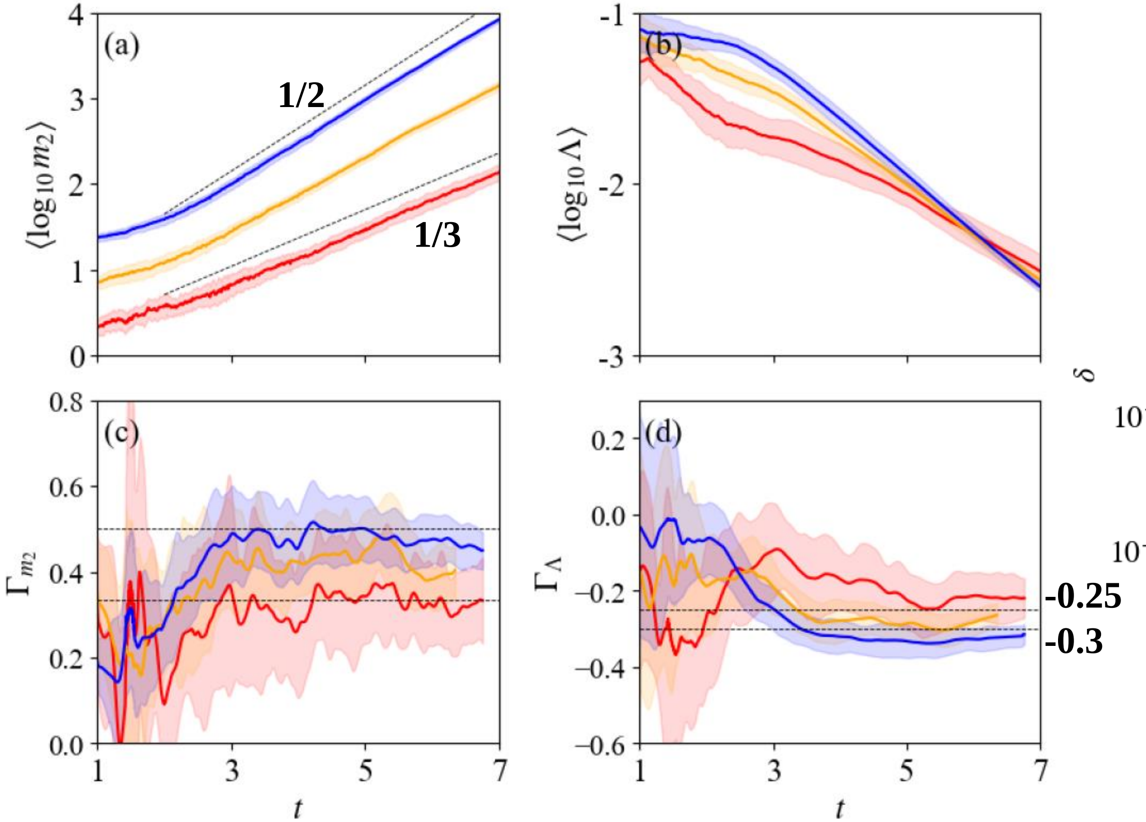


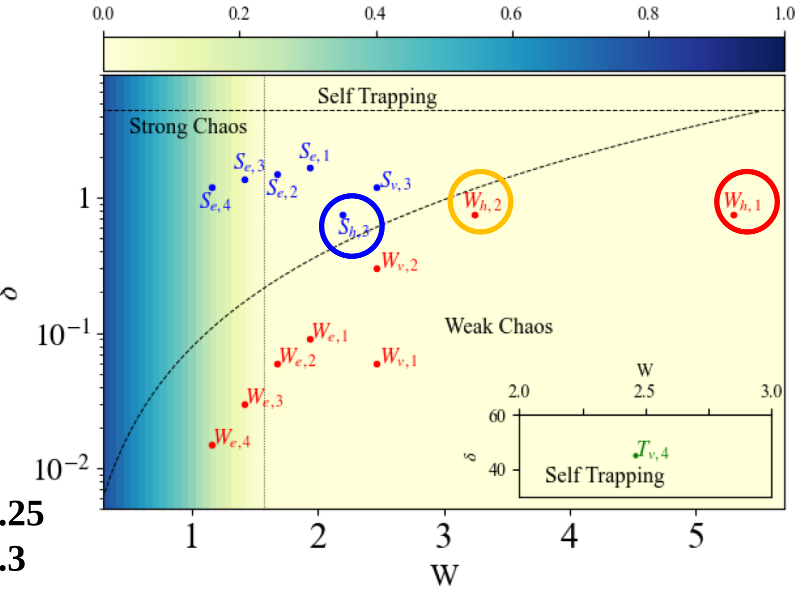
Global investigation II

$\beta = 0.25, W = 5.2, L = 3, N = 1001$

$\beta = 0.25, W = 3.5, L = 6, N = 1001$

$\beta = 0.25, W = 2.2, L = 15, N = 1001$





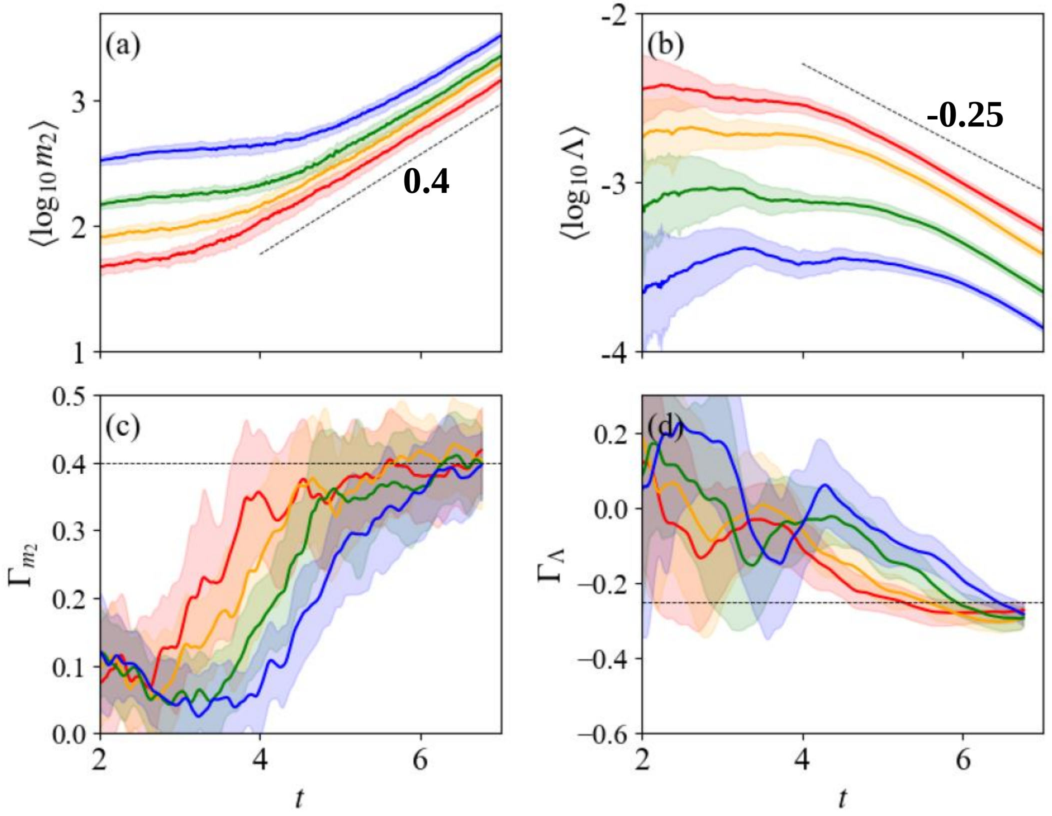
Effect of frequency gaps (Weak chaos)

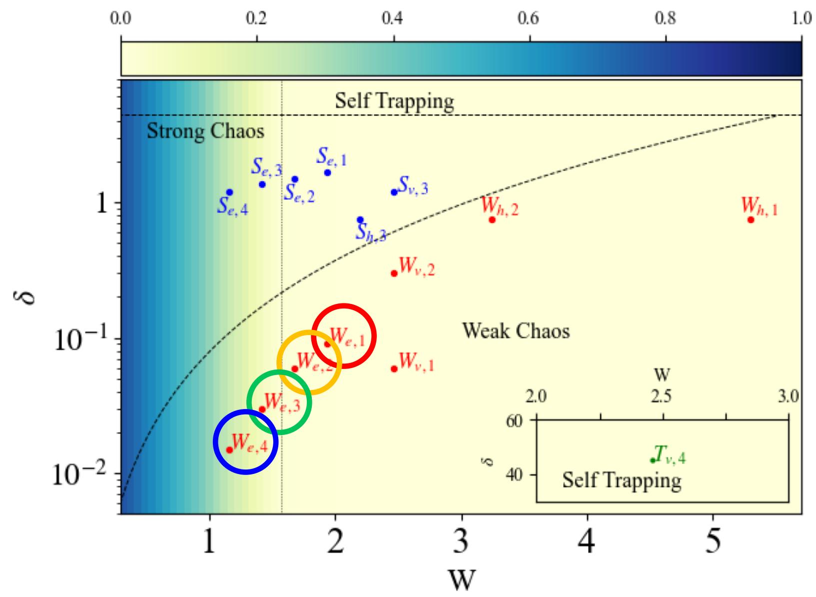
$\beta = 0.03, W = 1.94, L = 18, N = 1001, \alpha = 0$

$\beta = 0.02, W = 1.68, L = 24, N = 1001, \alpha = 0$

$\beta = 0.01, W = 1.42, L = 35, N = 1001, \alpha = 0.007$

$\beta = 0.005, W = 1.16, L = 53, N = 1001, \alpha = 0.124$





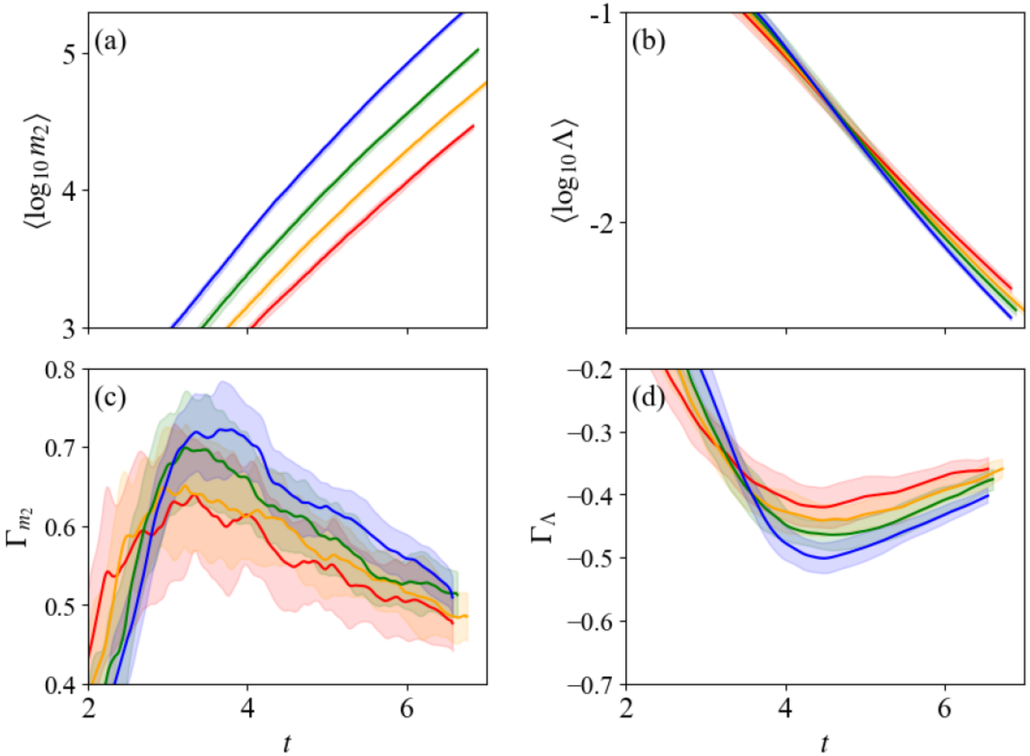
Effect of frequency gaps (Strong chaos)

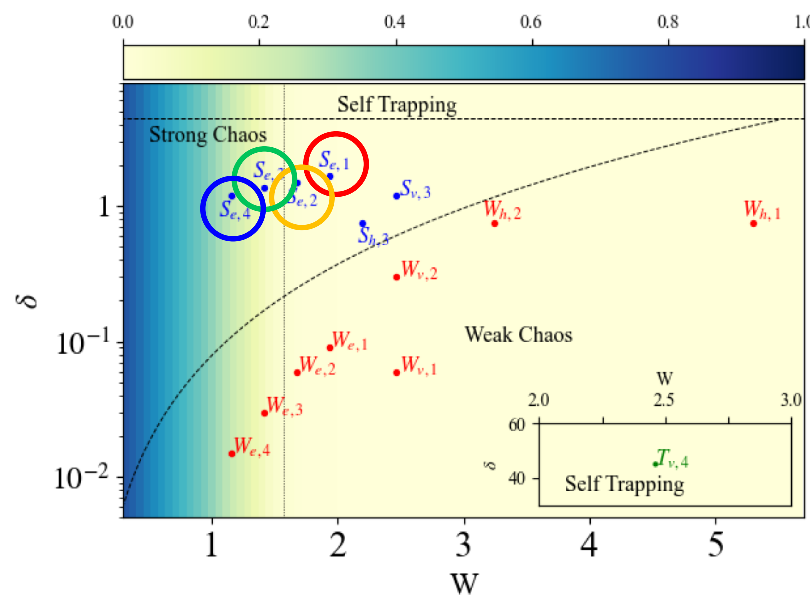
$\beta = 0.55, W = 1.94, L = 18, N = 1001, \alpha = 0$

$\beta = 0.5, W = 1.68, L = 24, N = 1001, \alpha = 0$

$\beta = 0.45, W = 1.42, L = 35, N = 1001, \alpha = 0.007$

$\beta = 0.4, W = 1.16, L = 53, N = 1001, \alpha = 0.124$





Summary

We investigated in detail the spatiotemporal chaotic behavior of the one-dimensional disordered nonlinear stub lattice model, whose linear counterpart exhibits a flat frequency band.

- **Identification of 2 different dynamical spreading regimes:**
  - **weak chaos regime:** $m_2 \sim t^{1/3}, P \sim t^{1/6}, \Lambda \sim t^{-0.25}$
  - **strong chaos regime:** $m_2 \sim t^{1/2}, P \sim t^{1/4}, \Lambda \sim t^{-0.3}$
- **The transition between the different dynamical regimes is not abrupt but happens in a rather smooth fashion.**
- **The presence of the frequency gap does not seem to strongly affect the dynamics of the weak chaos regime, while in the case of strong chaos further investigations are needed, as our results are rather inconclusive.**
- **The ftMLE reveals the decrease of the system's chaoticity in time.**
- **The DVDs provide information about the propagation of chaos.**

  Wandering of localized chaotic hot spots in the lattice's excited part homogenize chaos.

Main references

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